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2009 J. Phys. A: Math. Theor. 42 335202

(http://iopscience.iop.org/1751-8121/42/33/335202)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 335202 (12pp) doi:10.1088/1751-8113/42/33/335202

Nonautonomous integrable nonlinear Schrödinger equations with generalized external potentials

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Received 14 April 2009, in final form 22 June 2009
Published 28 July 2009
Online at stacks.iop.org/JPhysA/42/335202

Abstract
Inhomogeneous nonlinear Schrödinger equations are constructed systematically using the covariance with respect to the Darboux transformation. The Darboux invariants and the spectral parameter are functions of time and space, which result in equations describing nonautonomous solitons moving in generalized external potentials. All the known inhomogeneous nonlinear Schrödinger equations, as well as new ones of a more generalized form, can be derived from the constructed equation. One- and two-soliton solutions are explicitly constructed using the Darboux transformation.

PACS numbers: 42.65.Tg, 05.45.Yv

1. Introduction

Recently, much interest has arisen on the study of inhomogeneous nonlinear Schrödinger (NLS) equations [1–25]. They describe real physics problems occurring in the media of inhomogeneity and are more realistic compared to the standard NLS equation. Historically, the inhomogeneous NLS equation first appeared in [1], which studied the transmission of solitons through the varying dispersion-managed optical fiber. After that, various attempts have been made to construct and study the inhomogeneous NLS equations, including those having an integrable property and multi-soliton solutions [1–25]. These describe solitons in a nonuniform medium such as soliton lasers, soliton switches and matter-wave solitons of Bose–Einstein condensates with magnetically tuned interatomic interaction.

Nonautonomous NLS equations with generalized external potentials can be written as

$$\mathrm{i}\bar{\partial}\psi + k\partial^2\psi + 2a^2k|\psi|^2\psi + (\mathrm{i}\gamma + \delta)\partial\psi + (\mathrm{i}\Gamma + \Delta)\psi = 0, \tag{1}$$

where $\partial \equiv \partial/\partial z$ and $\bar{\partial} \equiv \partial/\partial\bar{z}$ with soliton time z and soliton moving distance $\bar{z}$ respectively. The coefficients $k, a, \gamma, \delta, \Gamma, \Delta$ are real functions of z and $\bar{z}$ describing inhomogeneities, but they need to be constrained appropriately for the equations to be integrable. The differential equations considered in this paper have $\bar{z}$-varying coefficients, and are called nonautonomous as they depend on the independent variable. In the physical context, the z-dependence of coefficients is connected with external potentials such as trapping or reflecting potentials. The NLS equation (1) has a complex form of coefficients, and is named as the nonautonomous equation with *generalized* external potentials. The present work can be thought of as an extended version of the nonautonomous solitons in external potentials dealt in [21].

In the literature, various specific forms of coefficients in equation (1) have been constructed, which have soliton or solitary solutions. Generally, the dispersion and nonlinearity coefficients have been considered as a function of $\bar{z}$ only [2]. Besides, the z-dependence of Δ has appeared to be limited for the polynomial type only. In this paper, we consider more generalized inhomogeneous NLS equations with both z- and $\bar{z}$-dependent coefficients. These types of equations can describe situations of physical interest, for example, in the field of Bose–Einstein condensates with spatial inhomogeneities and time-dependent potentials and nonlinearities [3]. For this purpose, we use the principle of Darboux covariance [26–31] to construct the Lax pair and the variable spectral parameter. The invariants of the Darboux transformation (DT) determine some matrix elements of the Lax pair, which were found to be effective in constructing nonlinear equations.

The principle of Darboux covariance has been used to study various integrable nonlinear equations, which include those associated with the $SU(2)$ linear system [26], $GL(2)$ system [27], time- and space-dependent Darboux invariants [28], its generalization to the (2 + 1)-dimensional system [29] and inhomogeneous systems with the variable spectral parameter [30, 31]. Recently a realistic program for constructing the inhomogeneous equations was introduced in [31], and was used to construct integrable equations having a specific form of inhomogeneity coefficients. However, the full implementation of this program for the NLS equations has not been achieved yet. In this paper, we will employ this program to construct the most generalized form of NLS equations allowed by the principle of Darboux covariance. The distinguishing feature of the present result compared to the previous ones [28, 31] is that the coefficients of equations are obtained from nine arbitrary functions, which will be called as the *inhomogeneity functions*. Previously, this type of functions has been constrained to each other, and solving these constraints was a difficult problem [31]. Only a specific form of coefficients of nonlinear equations has been obtained by solving the constraints for special cases. On the other hand, the nine *inhomogeneity functions* of this paper give coefficients of most generalized NLS equations allowed by the Darboux covariance principle.

We describe the Lax pair in section 2 and the spectral parameter in section 5. They give the most generalized form of NLS equations in section 3. This construction is based on the covariance principle with respect to the DT, whose explicit form is introduced in section 4. One- and two-soliton solutions are constructed in sections 6 and 7 for the most generalized equation with nine inhomogeneity functions. Some specific forms of coefficients are considered in section 8.

2. Lax pair

To construct equations with generalized coefficients, we consider a system having the variable spectral parameter $\lambda = \lambda(z, \bar{z})$. It satisfies certain relations of the following type, given by $\alpha_i = \alpha_i(z, \bar{z})$, $\beta_i = \beta_i(z, \bar{z})$, $i = 0, 1$:

$$\partial\lambda = \alpha_1\lambda + \alpha_0, \qquad \bar{\partial}\lambda = \beta_1\lambda + \beta_0. \tag{2}$$

The NLS equation is described by the associated linear equations:

$$\begin{aligned} 0 &= \partial\Phi + U(\lambda)\Phi \equiv \partial\Phi + (\lambda U_1 + U_0)\Phi, \\ 0 &= \bar{\partial}\Phi + V(\lambda)\Phi \equiv \bar{\partial}\Phi + (\lambda^2 V_2 + \lambda V_1 + V_0)\Phi. \end{aligned} \tag{3}$$

Some matrix elements of the Lax pair $U(\lambda)$, $V(\lambda)$ are related to the Darboux invariants and lead to the following form [28, 30, 31]:

$$U(\lambda) = \begin{pmatrix} \frac{\mathrm{i}}{2}(\lambda f + l) & a\,\mathrm{e}^{\mathrm{i}\theta}\psi \\ -a\,\mathrm{e}^{-\mathrm{i}\theta}\psi^* & -\frac{\mathrm{i}}{2}(\lambda f + l) \end{pmatrix} \tag{4}$$

and

$$V(\lambda) = \begin{pmatrix} \frac{\mathrm{i}}{2}(\lambda^2 f^2 k + \lambda h - 2ka^2|\psi|^2 + 2g) & \chi + \lambda f k a\,\mathrm{e}^{\mathrm{i}\theta}\psi \\ -\chi^* - \lambda f k a\,\mathrm{e}^{-\mathrm{i}\theta}\psi^* & -\frac{\mathrm{i}}{2}(\lambda^2 f^2 k + \lambda h - 2ka^2|\psi|^2 + 2g) \end{pmatrix}. \tag{5}$$

Here, ψ is the field variable of the NLS equation and

$$\begin{aligned} \chi &= \mathrm{i}\,\mathrm{e}^{\mathrm{i}\theta}\left[\left(\frac{a}{2}\partial k + k\partial a + \mathrm{i}ka\partial\theta - \mathrm{i}\frac{a}{f}h + \mathrm{i}lka\right)\psi + ka\partial\psi\right], \\ g &= \frac{1}{4}N - \frac{1}{8}b_1^2K^2 + \frac{1}{8}(\bar{\partial}b_1)K^2 + \frac{1}{8}\frac{h^2}{kf^2} - \frac{1}{4}\bar{\partial}H + \frac{1}{2}\bar{\partial}L + \frac{1}{2}b_0K + b_g, \end{aligned} \tag{6}$$

where

$$\begin{aligned} &K = K(z,\bar{z}) = \int \frac{1}{\sqrt{k}}\,\mathrm{d}z, \qquad P = P(z,\bar{z}) = \int \frac{\bar{\partial}K}{\sqrt{k}}\,\mathrm{d}z, \qquad N = N(z,\bar{z}) = \int \frac{\bar{\partial}^2K}{\sqrt{k}}\,\mathrm{d}z, \\ &H = H(z,\bar{z}) = \int \frac{h}{kf}\,\mathrm{d}z, \qquad L = L(z,\bar{z}) = \int l\,\mathrm{d}z, \end{aligned} \tag{7}$$

and $b_1 = b_1(\bar{z})$, $b_0 = b_0(\bar{z})$, $b_g = b_g(\bar{z})$ are arbitrary functions of $\bar{z}$. In equations (4) and (5), the Darboux invariants, $f = f(z,\bar{z})$, $l = l(z,\bar{z})$, $k = k(z,\bar{z})$, $h = h(z,\bar{z})$, are arbitrary real functions, which are introduced to satisfy the invariant conditions of DT imposed on U and V [28, 30, 31]. $a = a(z,\bar{z})$ and $\theta = \theta(z,\bar{z})$ are real, and are related to defining the field variable $\psi = \psi(z,\bar{z})$. An identity among P, K, N is useful in the following:

$$\bar{\partial}P = \bar{\partial}\int \bar{\partial}K\,\mathrm{d}K = \int \bar{\partial}^2K\,\mathrm{d}K + \int \bar{\partial}K\,\mathrm{d}(\bar{\partial}K) = N + \frac{1}{2}(\bar{\partial}K)^2. \tag{8}$$

In addition, α_i and β_i in equation (2) are given by

$$\begin{aligned} \alpha_1 &= -\frac{1}{2}\partial\ln(f^2k), \\ \alpha_0 &= \frac{-h\alpha_1 + f\beta_1 - \partial h + \bar{\partial}f}{2f^2k}, \\ \beta_1 &= -\frac{1}{2}\bar{\partial}\ln(f^2k) + b_1, \\ \beta_0 &= \frac{1}{2f\sqrt{k}}(\bar{\partial}^2K + \bar{\partial}b_1K - b_1^2K + 2b_0) - \frac{1}{2f^2k}\left(\bar{\partial}h - h\bar{\partial}\ln f - \frac{1}{2}h\bar{\partial}\ln k - b_1h\right). \end{aligned} \tag{9}$$

These nine *inhomogeneity functions* $f, l, k, h, a, \theta, b_g, b_0, b_1$ are real and arbitrary, which determine the Lax pair $U(\lambda)$, $V(\lambda)$ as well as the spectral parameter λ. They satisfy the compatibility conditions in sections 3 and 5, which guarantee the existence of Φ and λ, and lead to the integrable equation. By using specific forms of the inhomogeneity functions, various NLS equations with different types of coefficients and spectral parameters can be obtained; see section 8.

3. Inhomogeneous NLS equations

The compatibility of the Lax pair in equation (3), i.e. $\partial\bar{\partial}\Phi = -\partial(V\Phi) = \bar{\partial}\partial\Phi = -\bar{\partial}U\Phi$, gives the NLS equation of motion (1) with

$$
\begin{aligned}
\delta &= \frac{3}{2}\partial k + 2k\partial\ln a,\\
\gamma &= 2k\partial\theta + 2kl - \frac{h}{f},\\
\Delta &= \frac{1}{2}\partial^2 k - kl^2 - k(\partial\theta)^2 - 2kl\partial\theta + \frac{hl}{f} + \frac{3}{2}\partial k\partial\ln a + \frac{h\partial\theta}{f} - \frac{1}{4}\frac{h^2}{kf^2} + \frac{1}{4}b_1^2K^2\\
&\quad - b_0K + \frac{1}{2}\bar{\partial}H - \bar{\partial}L - 2b_g - \frac{1}{4}K^2\bar{\partial}b_1 - \frac{1}{2}N + \frac{k\partial^2 a}{a} - \bar{\partial}\theta,\\
\Gamma &= 2kl\partial\ln a - \frac{h}{f}\partial\ln a + 2k\partial\theta\partial\ln a + \bar{\partial}\ln a + \frac{3}{2}l\partial k + \frac{3}{2}\partial\theta\partial k + \frac{1}{2}\frac{h}{f}\partial\ln f\\
&\quad - \frac{1}{2}\frac{\partial h}{f} - \frac{1}{4}\frac{h}{f}\partial\ln k + \frac{1}{4}\bar{\partial}\ln k + k\partial l + k\partial^2\theta - \frac{1}{2}b_1.
\end{aligned}
\tag{10}
$$

We note that the equation resulting from the compatibility of the Lax pair does not contain the spectral parameter λ, which ensures the integrability of the inhomogeneous NLS equation with equation (10). This is the main formula of this paper, giving a generalized form of coefficients in equation (10) based on the principle of Darboux covariance. Previous studies on inhomogeneous NLS equations [1–25] have introduced coefficients such that k, a and λ were taken to be functions of $\bar{z}$ only. Only Δ was seen to have z-dependence, though its dependence is limited to the polynomial form. Though the coefficients has been of a realistic form with wide applicability, more generalized coefficients with both z- and $\bar{z}$-dependences should extend the application spectrum of the inhomogeneous NLS equation. In the present study, various forms of inhomogeneity coefficients will be obtained by taking appropriate *inhomogeneity functions* $k, l, h, f, a, \theta, b_1, b_0, b_g$.

4. Darboux transformation

The Lax pair was shown to be consistent with the following DT [32, 33]:

$$
\Phi \to \Phi^{[N]} = S(\lambda,\lambda_1)\left[\lambda - \lambda_1^* - (\lambda_1 - \lambda_1^*)\frac{\Phi_1\Phi_1^\dagger}{\Phi_1^\dagger\Phi_1}\right]\Phi \equiv S(\lambda,\lambda_1)[\lambda-\sigma]\Phi, \tag{11}
$$

where the two-component column matrix Φ_1 is a solution to equation (3) at a specific value of $\lambda = \lambda_1$, and

$$
S(\lambda,\lambda_1) = (\lambda^2 - (\lambda_1+\lambda_1^*)\lambda + \lambda_1\lambda_1^*)^{-1/2}. \tag{12}
$$

It can be explicitly checked that f, l, k, h in U and V are Darboux invariants, and they are consistent with the DT in equation (11).

Under the DT, ψ transforms to a new field $\psi^{[N]}$, which can be obtained by considering the transformation property of U_0:

$$
U_0^{[N]} = \begin{pmatrix} \frac{\mathrm{i}}{2}l & a\,\mathrm{e}^{\mathrm{i}\theta}\psi^{[N]} \\ a\,\mathrm{e}^{-\mathrm{i}\theta}\psi^{[N]*} & -\frac{\mathrm{i}}{2}l \end{pmatrix} = U_0 + [U_1,\sigma], \tag{13}
$$

such that

$$
\psi^{[N]} = \psi + \mathrm{i}\frac{f}{a}\,\mathrm{e}^{-\mathrm{i}\theta}\sigma_{1,2}. \tag{14}
$$

This formula will be used to construct multi-soliton solutions in sections 6 and 7.

5. Spectral parameter

It can be explicitly checked that α_i, β_i in equation (9) satisfy the compatibility equation, $\partial\bar{\partial}\lambda = \partial(\beta_1\lambda + \beta_0) = \bar{\partial}\partial\lambda = \bar{\partial}(\alpha_1\lambda + \alpha_0)$, which ensures the existence of $\lambda = \lambda(z, \bar{z})$. Explicitly, α_i, β_i in equation (9) satisfy

$$\bar{\partial}\alpha_1 = \partial\beta_1 + \alpha_1\beta_1, \qquad \bar{\partial}\alpha_0 + \alpha_1\beta_0 = \partial\beta_0 + \alpha_0\beta_1. \tag{15}$$

Equation (2) can be integrated to give the variable spectral parameter

$$\lambda = -\frac{1}{2}\frac{h}{f^2k} + \frac{1}{2}\frac{1}{f\sqrt{k}}(\bar{\partial}K + b_1K + 2\Lambda), \tag{16}$$

where

$$\Lambda = \Lambda(zb) = \left(\int b_0\,\mathrm{e}^{-\int b_1\,\mathrm{d}\bar{z}}\,\mathrm{d}\bar{z} + \mu\right)\mathrm{e}^{\int b_1\,\mathrm{d}\bar{z}}, \tag{17}$$

where μ is the so-called hidden spectral parameter [34–36].

6. One-soliton

Having formulated the NLS equation as the compatibility condition of the linear equations (3), we can use the DT in section 4 to generate a new solution from a known one. To obtain the one-soliton solution, we start with $\psi = 0$. Then, the solution of linear equations in equation (3) becomes

$$\begin{aligned}\Phi &= \exp\left[\frac{\mathrm{i}}{2}\left(\frac{H}{2} - \frac{P}{2} - \frac{b_1}{4}K^2 - \Lambda K - L - \int(\Lambda^2 + 2b_g)\,\mathrm{d}\bar{z}\right)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right]\Phi_0\\ &\equiv \exp\left[\frac{\mathrm{i}}{2}\Xi\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right]\Phi_0,\end{aligned} \tag{18}$$

where Φ_0 is a constant matrix. Φ_1 in equation (11) is obtained from equation (18) by taking $\Lambda = \Lambda_1$, where Λ_1 is obtained by inserting a specific complex value $\mu = \mu_1$ ($\mu_1 = \mu_{1r} + \mathrm{i}\mu_{1i}$) into Λ in equation (16). All functions $(H, P, b_0, b_1, \ldots)$ in Φ_1 are real-valued, except Λ_1 which is $\Lambda_1 = \Lambda_{1r} + \mathrm{i}\Lambda_{1i}$. Then Ξ_1 in equation (18) becomes complex-valued such that $\Xi_1 = \Xi_{1r} + \mathrm{i}\Xi_{1i}$ with

$$\begin{aligned}\Xi_{1r} &= \frac{H}{2} - \frac{P}{2} - \frac{b_1}{4}K^2 - \Lambda_{1r}K - L - \int\left(\Lambda_{1r}^2 - \Lambda_{1i}^2 + 2b_g\right)\mathrm{d}\bar{z},\\ \Xi_{1i} &= -\Lambda_{1i}K - 2\int\Lambda_{1r}\Lambda_{1i}\,\mathrm{d}\bar{z},\end{aligned} \tag{19}$$

and (we take $\Phi_0 = \binom{1}{1}$ for simplicity)

$$\Phi_1 = \Phi|_{\mu=\mu_1} = \exp\left[\frac{1}{2}(-\Xi_{1i} + \mathrm{i}\Xi_{1r})\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right]\Phi_0 = \begin{pmatrix}\exp\left[\frac{1}{2}(-\Xi_{1i} + \mathrm{i}\Xi_{1r})\right]\\ \exp\left[-\frac{1}{2}(-\Xi_{1i} + \mathrm{i}\Xi_{1r})\right]\end{pmatrix}. \tag{20}$$

By using the definition of σ in equation (11) and utilizing Φ_1, we obtain

$$\sigma = \lambda_1^* + \frac{1}{2}(\lambda_1 - \lambda_1^*)\operatorname{sech}\Xi_{1i}\begin{pmatrix}\exp(-\Xi_{1i}) & \exp(\mathrm{i}\Xi_{1r})\\ \exp(-\mathrm{i}\Xi_{1r}) & \exp(\Xi_{1i})\end{pmatrix}, \tag{21}$$

where λ_1 is obtained from equation (16) by substituting $\Lambda = \Lambda_1$.

Finally, equation (14) gives the one-soliton solution $\psi^{(1)}$:

$$\psi^{(1)}(z, \bar{z}) = \psi^{[N]} = -\frac{\mu_{1i}}{a\sqrt{k}}\operatorname{sech}\Xi_{1i}\exp\left(\mathrm{i}\Xi_{1r} - \mathrm{i}\theta + \int b_1\,\mathrm{d}\bar{z}\right). \tag{22}$$

We explicitly check that equation (22) satisfies the NLS equation (1) with (10) using the symbolic package 'MAPLE'. The one-soliton solution of the inhomogeneous NLS equation shares some important properties of the soliton solution with the standard NLS equation such that it is stable and survives the collision between them; see section 7.

7. Two-soliton

To calculate the two-soliton solution, we need a solution $\Phi^{[N]}$ of equation (3) with $\psi = \psi^{(1)}$ to obtain $\Phi_2 = \Phi^{[N]}_{\mu=\mu_2}$. The DT in equation (11) gives the required solution such that

$$\Phi_2 = \Phi^{[N]}|_{\mu=\mu_2=\mu_{2r}+i\mu_{2i}} = \left[\lambda_2 - \lambda_1^* - (\lambda_1 - \lambda_1^*)\frac{\Phi_1\Phi_1^\dagger}{\Phi_1^\dagger\Phi_1}\right]\Phi|_{\mu=\mu_2}, \tag{23}$$

where λ_2 is given by equations (16) and (17) with $\mu = \mu_2 = \mu_{2r} + \mathrm{i}\mu_{2i}$. Then the two-soliton solution $\psi^{(2)}$ is given by equation (14) [4, 9, 17, 28]:

$$\psi^{(2)} = \psi^{(1)} + \mathrm{i}\frac{f}{a}\,\mathrm{e}^{-\mathrm{i}\theta}(\lambda_2 - \lambda_2^*)\left(\frac{\Phi_2\Phi_2^\dagger}{\Phi_2^\dagger\Phi_2}\right)_{12}. \tag{24}$$

With Ξ_{1r}, Ξ_{1i} in equation (19) and similarly defined Ξ_{2r}, Ξ_{2i} (using μ_2 instead of μ_1 in Ξ_{1r}, Ξ_{1i}), we can calculate the two-soliton solution such that

$$\psi^{(2)} = \frac{\mathrm{i}}{a\sqrt{k}}\,\mathrm{e}^{-\mathrm{i}\theta+\int b_1\,\mathrm{d}\bar{z}}\,\frac{(A\sinh\Xi_{2i} + \mathrm{i}B\cosh\Xi_{2i})\,\mathrm{e}^{\mathrm{i}\Xi_{1r}} - (A\sinh\Xi_{1i} + \mathrm{i}C\cosh\Xi_{1i})\,\mathrm{e}^{\mathrm{i}\Xi_{2r}}}{D\cosh\Xi_{1i}\cosh\Xi_{2i} - 2\mu_{1i}\mu_{2i}[\sinh\Xi_{1i}\sinh\Xi_{2i} + \cos(\Xi_{2r} - \Xi_{1r})]}, \tag{25}$$

where

$$\begin{aligned} &A = 2\mu_{1i}\mu_{2i}(\mu_{2r} - \mu_{1r}), \qquad B = \mu_{1i}\left[\mu_{1i}^2 + (\mu_{2r} - \mu_{1r})^2 - \mu_{2i}^2\right],\\ &C = \mu_{2i}\left[\mu_{1i}^2 - (\mu_{2r} - \mu_{1r})^2 - \mu_{2i}^2\right], \qquad D = \mu_{1i}^2 + (\mu_{2r} - \mu_{1r})^2 + \mu_{2i}^2. \end{aligned} \tag{26}$$

8. Special cases

8.1. Case of $\delta = \gamma = 0$

First, we introduce a set of the inhomogeneity functions, which results in an inhomogeneous NLS equation without the $\partial\psi$ term. For this, we take

$$a = \frac{b_a(\bar{z})}{k(z,\bar{z})^{3/4}}, \qquad \theta = \frac{1}{2}H - L + b_\theta(\bar{z}), \tag{27}$$

where $b_a(\bar{z})$ and $b_\theta(\bar{z})$ are arbitrary functions. Then, equation (10) gives $\delta = \gamma = 0$ and

$$\begin{aligned} &\Delta = \frac{1}{4}b_1^2K^2 - b_0K - 2b_g - \frac{1}{4}\bar{\partial}b_1K^2 - \frac{1}{2}N + \frac{3}{16}\frac{(\partial k)^2}{k} - \bar{\partial}b_\theta - \frac{1}{4}\partial^2 k,\\ &\Gamma = \bar{\partial}\ln b_a - \frac{1}{2}b_1 - \frac{1}{2}\bar{\partial}\ln k. \end{aligned} \tag{28}$$

Equation (1) with these coefficients is an inhomogeneous NLS equation, which was widely studied to describe physical systems with pumping or attenuation effects ($\Gamma \neq 0$) or systems lying in a potential ($\Delta \neq 0$). We note that various forms of $k(z,\bar{z})$ are possible. For example, taking $k(z,\bar{z}) = k_1(z)k_2(\bar{z})$ results in $\Xi_i = -\Lambda_i k_2^{-1/2}\int k_1^{-1/2}\,\mathrm{d}z - 2\int\Lambda_r\Lambda_i\,\mathrm{d}\bar{z}$, which gives a nontrivial z-dependence of soliton solutions.

8.2. Case of $\delta = \gamma = 0, k(z,\bar{z}) = k_2(\bar{z})$

In the literature, there appear NLS equations where k depends on $k_2(\bar{z})$ only. (In fact, the general case of $k = k(z,\bar{z})$ has not appeared.) They are related to the integrable Gross–Pitaevskii equation with time dependences and/or with the Feshbach resonance management [5]. In this case, they are described by following equation:

$$\mathrm{i}\bar{\partial}\psi + k_2\partial^2\psi + 2\frac{b_a^2}{\sqrt{k_2}}|\psi|^2\psi + \mathrm{i}\left(-\frac{1}{2}\bar{\partial}\ln k_2 - \frac{1}{2}b_1 + \bar{\partial}\ln b_a\right)\psi + \Delta\psi = 0, \tag{29}$$

where

$$\Delta = \left(-\frac{1}{16}(\bar{\partial}\ln k_2)^2 + \frac{1}{8}\bar{\partial}^2\ln k_2 + \frac{1}{4}b_1^2 - \frac{1}{4}\bar{\partial}b_1\right)z^2 - \frac{b_0}{\sqrt{k_2}}z - 2b_g - \bar{\partial}b_\theta. \tag{30}$$

Here $k_2, b_0, b_1, b_g, b_a, b_\theta$ are arbitrary functions of $\bar{z}$. Radha *et al* [4] studied an NLS equation with $b_g = -\bar{\partial}b_\theta/2$ such that no z^0-term appears in Δ.

8.3. Case of $\delta = \gamma = 0$, z^2-dependent Δ

A more specialized form of equation (29), as seen in the following, was frequently studied, especially by using the similariton approach [6–11]:

$$\mathrm{i}\bar{\partial}\psi = -\frac{k_2}{2}\partial^2\psi + \hat{\gamma}|\psi|^2\psi + \frac{1}{2}\hat{M}z^2\psi + \frac{\mathrm{i}}{2}(\hat{g} + \bar{\partial}\ln k_2)\psi, \tag{31}$$

where $\hat{g} = \hat{g}(\bar{z})$ and

$$\hat{\gamma} = -\left(\frac{8}{k_2}\right)^{1/2}\exp\left(-\int(\hat{g} - b_1)\,\mathrm{d}\bar{z}\right), \qquad \hat{M} = \frac{1}{k_2}\left(\bar{\partial}b_1 - b_1^2 - \frac{1}{2}\frac{\bar{\partial}^2k_2}{k_2} + \frac{3}{4}\frac{(\bar{\partial}k_2)^2}{k_2^2}\right). \tag{32}$$

This case corresponds to taking the following inhomogeneity functions in addition to equation (27):

$$k(z,\bar{z}) = k_2(\bar{z})/2, \qquad b_g = -\bar{\partial}b_\theta/2, \qquad b_0 = 0, \qquad b_a = \exp\left(-\int(\hat{g} - b_1)\,\mathrm{d}\bar{z}/2\right). \tag{33}$$

The equation describes a physical system having inhomogeneous dispersion.

A one-soliton solution can be obtained from equations (22) and (19) by substituting the inhomogeneity functions in equation (33) such that

$$\psi^{(1)} = -\frac{k_2}{2}^{(1/4)}\mu_{1i}\hat{A}^{(1/2)}\mathrm{sech}\left[\sqrt{\frac{2}{k_2}}\mu_{1i}\hat{A}(z - \hat{l})\right]\exp\left(\mathrm{i}\hat{\Phi} + \int\hat{g}\,\mathrm{d}\bar{z}/2\right), \tag{34}$$

where

$$\begin{aligned}\hat{A} &= \exp\left(\int b_1\,\mathrm{d}\bar{z}\right), \qquad \hat{l} = -\sqrt{2k_2}\mu_{1r}\int\hat{A}^2\,\mathrm{d}\bar{z}/\hat{A},\\ \hat{\Phi} &= -\frac{1}{2k_2}\left(b_1 - \frac{1}{2}\bar{\partial}\ln k_2\right)z^2 - \sqrt{\frac{2}{k_2}}\mu_{1r}\hat{A}z + \left(\mu_{1i}^2 - \mu_{1r}^2\right)\int\hat{A}^2\,\mathrm{d}\bar{z}.\end{aligned} \tag{35}$$

It is interesting to compare this solution with that obtained by the similariton approach. The one-soliton solution in [6–8] corresponds to the special case of $k_2 = 1$ in equations (34) and (35). In this case, the functions $\hat{g}, \hat{M}, \hat{A}, \hat{l}, \hat{\Phi}$ satisfy exactly the same relation in equations (5)–(8) of [6]. It thus means that equation (31) with $k_2 = 1$ is in fact integrable, though it was obtained by the similariton approach.

8.4. Case of $\delta = \gamma = \Delta = 0$

Many of the similariton approaches [12–16] treat the case of $\delta = \gamma = \Delta = 0$. Finding a general solution for $k(z, \bar{z})$, which satisfies the condition $\Delta = 0$ in equation (28), is difficult. In this paper, we treat a simple case of $k(z, \bar{z}) = k_1(z)k_2(\bar{z})$. Then, k_1 should satisfy certain differential equations, which is found to have two cases.

Case (1). $k_1(z) \equiv s(z)^2$ satisfies (α is an arbitrary constant)

$$\frac{\mathrm{d}^3 s(z)}{\mathrm{d}z^3} = \frac{\alpha}{2s(z)^2}. \tag{36}$$

In this case, we should take (in addition to equation (27))

$$b_1 = \frac{1}{2}\frac{\bar{\partial} k_2}{k_2} + \frac{c_0 k_2}{1 - c_0 \int k_2 \,\mathrm{d}\bar{z}}, \qquad b_0 = -\frac{\alpha}{4} k_2^{3/2}, \qquad b_g = -\frac{1}{2}\bar{\partial} b_\theta. \tag{37}$$

This inhomogeneity function gives an equation

$$\mathrm{i}\bar{\partial}\psi + s^2 k_2 \partial^2 \psi + 2\frac{b_a^2}{s k_2^{1/2}}|\psi|^2\psi + \mathrm{i}\Gamma\psi = 0, \tag{38}$$

where

$$\Gamma = \bar{\partial}\left[-\frac{3}{4}\ln k_2 + \frac{1}{2}\ln\left\{1 - c_0 \int k_2 \,\mathrm{d}\bar{z}\right\} + \ln b_a\right]. \tag{39}$$

The coefficient Γ is the same type found in [12], but the dispersion and nonlinearity coefficients now have both z- and $\bar{z}$-dependences. The NLS equation in [12, 16–20] can be obtained by taking $\alpha = 0$ and $s(z) = 1$ in equation (38).

Case (2). $k_1(z) \equiv s(z)^2$ satisfies (α is an arbitrary constant)

$$\frac{\mathrm{d}^3 s(z)}{\mathrm{d}z^3} = -2\frac{\alpha}{s(z)^2}\sqrt{\left(\frac{\mathrm{d}s(z)}{\mathrm{d}z}\right)^2 - 2s(z)\frac{\mathrm{d}^2 s(z)}{\mathrm{d}z^2}}. \tag{40}$$

In these cases, b_1, b_g are taken as in equation (37), while $b_0 = 0$. This case gives an NLS equation in (38) with

$$\Gamma = \bar{\partial}\left[-\frac{3}{4}\ln k_2 + \ln b_a\right] + \alpha k_2 \tan\left(c - 2\alpha \int k_2 \,\mathrm{d}\bar{z}\right). \tag{41}$$

When we take $c = \pi/2 + 2\alpha/c_0, \alpha \to 0$, equation (41) reduces to equation (39).

We note that in both cases of (1) and (2), the $\alpha = 0$ limit corresponds to taking $k_1(z) = (d_1 z^2 + 2d_2 z + d_3)^2$. In this case, the one-soliton solution is

$$\psi^{(1)} = -\left(k_1 k_2^3\right)^{1/4} \frac{\mu_{1i}}{b_a\left(1 - c_0 \int k_2 \,\mathrm{d}\bar{z}\right)} \operatorname{sech}\left[\frac{\mu_{1i}}{1 - c_0 \int k_2 \,\mathrm{d}\bar{z}}\left(Z + 2\mu_{1r}\int k_2 \,\mathrm{d}\bar{z}\right)\right]$$
$$\times \exp\left[\frac{\mathrm{i}}{1 - c_0 \int k_2 \,\mathrm{d}\bar{z}}\left(-\frac{c_0}{4}Z^2 - \mu_{1r} Z + \left(\mu_{1i}^2 - \mu_{1r}^2\right)\int k_2 \,\mathrm{d}\bar{z}\right)\right], \tag{42}$$

where

$$Z = \frac{1}{\sqrt{d_1 d_3 - d_2^2}} \tan^{-1} \frac{d_1 z + d_2}{\sqrt{d_1 d_3 - d_2^2}}. \tag{43}$$

Z reduces to z for $d_2 = 0, d_3 = 1, d_1 \to 0$. In obtaining the above expressions, we take various integral constants of K, P, H, L, N in equation (7) as zero. The one-soliton solution in [15] is different from equation (42) to the extent of these constants.

8.5. Case of $\delta = \gamma = \Gamma = 0$

To have $\Gamma = 0$ in equation (28), we need (in addition to equation (27))

$$k(z,\bar{z}) = k_1(z)k_2(\bar{z}), \qquad b_1 = 2\bar{\partial}\ln b_a - \bar{\partial}\ln k_2. \tag{44}$$

In this case, $k_1(z), k_2(\bar{z})$ can be chosen arbitrarily, and the NLS equation becomes

$$\mathrm{i}\bar{\partial}\psi + k_1(z)k_2(\bar{z})\partial^2\psi + 2\frac{b_a^2}{\sqrt{k_1(z)k_2(\bar{z})}}|\psi|^2\psi + \Delta\psi = 0, \tag{45}$$

with

$$\begin{aligned}\Delta = \frac{1}{k_2}\left(\frac{3}{16}(\bar{\partial}\ln k_2)^2 + \frac{3}{8}\bar{\partial}^2\ln k_2 - \frac{1}{2}\bar{\partial}^2\ln b_a + (\bar{\partial}\ln b_a)^2 - \bar{\partial}\ln b_a\bar{\partial}\ln k_2\right)\left(\int\frac{1}{\sqrt{k_1}}\,\mathrm{d}z\right)^2 \\ - \frac{b_0}{\sqrt{k_2}}\left(\int\frac{1}{\sqrt{k_1}}\,\mathrm{d}z\right) - 2b_g - \frac{1}{4}k_2\partial^2 k_1 + \frac{3k_2}{16k_1}(\partial k_1)^2 - \bar{\partial}b_\theta.\end{aligned} \tag{46}$$

Especially, a special case of $k_1 = 1$ in equations (45) and (46) gives the NLS equation and one-soliton solution of [21]. By substituting $f = 1, h = (\bar{\partial}K + b_1 K)\sqrt{k}$ in equation (16), we obtain the spectral parameter introduced in [21]. We note that f, h do not appear in the NLS equation (45) and in the one- and two-soliton solutions either. It shows that a different choice of f, h results in the same system in this case. In fact, the relation between θ and H in equation (27) results in a cancellation of f, h in γ, Δ, Γ in (10).

Results of [22] can be obtained by taking $k_1 = k_2 = 1, b_\theta = 0, b_a^2 = c\,\mathrm{e}^{q\bar{z}}/(\mathrm{d}\,\mathrm{e}^{2q\bar{z}} - c')$. This inhomogeneity function gives an NLS equation

$$\mathrm{i}\bar{\partial}\psi + \partial^2\psi + 2\frac{c\,\mathrm{e}^{q\bar{z}}}{d\,\mathrm{e}^{2q\bar{z}} - c'}|\psi|^2\psi + \left[\frac{1}{4}q^2z^2 - b_0 z - 2b_g\right]\psi = 0. \tag{47}$$

Here, b_0 and b_g are arbitrary $\bar{z}$-dependent functions, though Khawaja [22] treats them as constants. Khawaja [22] also considers the case $b_a^2 = c/(c\bar{z} + c')$, which gives an NLS equation without a z^2-dependent term. A one-soliton solution lying on a continuous wave was studied in [23] for this type of NLS equation. Similarly, results of [24] can be obtained by taking $k_1 = k_2 = b_a = 1, b_\theta = b_g = 0$.

8.6. Case of $\delta = \Delta = 0$

There exist many possibilities on inhomogeneity functions which give NLS equations with $\delta = \Delta = 0$. Here, we confine ourselves to the simple case of $f(z,\bar{z}) = 1, l(z,\bar{z}) = 0, h(z,\bar{z}) = h_1(z)h_2(\bar{z}), k(z,\bar{z}) = k_1(z)k_2(\bar{z}), \theta(z,\bar{z}) = \theta_1(z)\theta_2(\bar{z}), b_1(\bar{z}) = (\bar{\partial}\ln k_2)/2$. To have $\delta = 0$, we need to take $a(z,\bar{z}) = b_a(\bar{z})/[k_1(z)k_2(\bar{z})]^{3/4}$. $\Delta = 0$ requires more stringent conditions on $k_1, k_2, h_1, h_2, \theta_1, \theta_2, b_0, b_g$. We consider four cases of k_1, h_1 here.

Case (1). $k_1 = h_1 = 1$

In this case, we take inhomogeneity functions $b_g, b_0, \theta_1, \theta_2$ as

$$\begin{aligned} b_g &= -\frac{1}{8}\frac{h_2^2}{k_2}, \qquad & b_0 &= -\frac{1}{2}\frac{h_2\bar{\partial}k_2}{k_2^{3/2}} + \frac{1}{2}\frac{\bar{\partial}h_2}{\sqrt{k_2}} + \frac{1}{2}\frac{h_2}{\sqrt{k_2}\left(\int k_2\,\mathrm{d}\bar{z} + c\right)}, \\ \theta_1 &= z^2, & \theta_2 &= \frac{1}{4}\frac{1}{\int k_2\,\mathrm{d}\bar{z} + c}, \end{aligned} \tag{48}$$

while k_2, h_2 are arbitrary. These inhomogeneity functions give an NLS equation,

$$\mathrm{i}\bar{\partial}\psi + k_2\partial^2\psi + \mathrm{i}\left(z\bar{\partial}\ln\left(\int k_2\,\mathrm{d}\bar{z} + c\right) - h_2\right)\partial\psi + 2\frac{b_a^2}{\sqrt{k_2}}|\psi|^2\psi$$
$$+\mathrm{i}\bar{\partial}\left(\frac{1}{2}\ln\left(\int k_2\,\mathrm{d}\bar{z} + c\right) + \ln b_a - \frac{3}{4}\ln k_2\right)\psi = 0. \quad (49)$$

This equation was studied in [21, 25].

Case (2). $k_1 = 1, h_1 = c_1 - z$

Case (3). $k_1 = z^2, h_1 = c_1 z - z\ln z$

Case (4). $k_1 = z^4, h_1 = c_1 z^2 + z$

In cases of (2), (3) and (4), we take inhomogeneity functions as $b_g = b_0 = \theta_1 = \theta_2 = 0, h_2 = \bar{\partial}\ln(\int k_2\,\mathrm{d}\bar{z} + c)$, while k_2 is arbitrary. These cases give the following equation (with appropriate k_1, h_1 for each case), which have z-dependent dispersion and nonlinearity:

$$\mathrm{i}\bar{\partial}\psi + k_1k_2\partial^2\psi - \mathrm{i}h_1\bar{\partial}\ln\left(\int k_2\,\mathrm{d}\bar{z} + c\right)\partial\psi + 2\frac{b_a^2}{\sqrt{k_1k_2}}|\psi|^2\psi$$
$$+\mathrm{i}\bar{\partial}\left(\frac{1}{2}(h_1\partial\ln k_1 - \partial h_1)\ln\left(\int k_2\,\mathrm{d}\bar{z} + c\right) + \ln b_a - \frac{3}{4}\ln k_2\right)\psi = 0. \quad (50)$$

9. Discussion

In this paper, we study nonautonomous NLS equations with coefficients of both time and space dependences by constructing the Lax pair using the principle of Darboux covariance. A similar study on an NLS equation has appeared [3], but that equation was not integrable. The most interesting feature of the present study is that there are nine arbitrary inhomogeneity functions. These inhomogeneity functions satisfy the compatibility condition of the Lax pair, which guarantee the existence of Φ in equation (3). They also satisfy the compatibility condition in equation (15) leading to the spectral parameter in equation (16). The DT allows us to write down N-soliton solutions (possibly in a matrix determinant form; see for example [37–42]). One- and two-soliton solutions for equations with time- and space-modulated coefficients are explicitly constructed.

This paper only considers the case of $\beta_2 = 0$ in equation (2). A more general type of the coupling term, $\int \beta_2|\psi|^2\,\mathrm{d}z\psi$, results when we consider a spectral parameter described by $\beta_2 \neq 0$ [31]. We do not consider this possibility, because this type of coupling term is not related directly to the physical systems of inhomogeneous fiber or the Bose–Einstein condensates in a potential.

The nonlinear NLS equations presented in this paper are constructed using the principle of Darboux covariance and are thus integrable. The integrability allows us to construct various types of solutions including the cnoidal waves, the N-phase waves and solitons lying on a cnoidal wave [43].

The generalized higher order NLS equation with variable coefficients is considered in [19, 44–46] as a soliton control system. Similarly, a derivative nonlinear Schrödinger equation with variable coefficients is studied in [47]. Our approach can be generalized to construct these types of equations with more generalized coefficients of time and space modulations. Another generalization of the NLS equation is the construction of multi-component equations. An important development of multi-component equations has been based on group theoretical extension, and finds many interesting physical applications as in [32, 33, 45, 48, 49]. It would be interesting to apply the present approach to multi-component NLS equations of the inhomogeneous type.

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